CENTER CYCLICITY OF LORENZ, CHEN AND LÜ SYSTEMS

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Abstract. This work provides upper bounds on the cyclicity of the centers on center manifolds in the well-known Lorenz family, and also in the Chen and Lü families. We prove that at most one limit cycle can be made to bifurcate from any center of any element of these families, perturbing within the respective family, with the exception of one specific Lorenz system where the cyclicity increases. We also show that this bound is sharp.

1. Introduction

In [19] Edward Lorenz developed, as a model of atmospheric convection, the celebrated system of three ordinary differential equations that bear his name,

$$\dot{x} = \sigma(y - x), \quad \dot{y} = \rho x - y - xz, \quad \dot{z} = -bz + xy, \tag{1}$$

where σ and ρ, respectively, are proportional to the Prandtl and Rayleigh numbers and b is the aspect ratio of the convection cylinders. System (1) is derived by simplifying the Navier-Stokes equations describing thermal convection in a viscous liquid film under a gradient of temperatures between the upper and lower surfaces. The physical interest of the Lorenz system is usually restricted to positive parameter values. However, in some cases this restriction can be broken as, for example, in the study of a convection model [17] and in the analysis of traveling-wave solutions of the Maxwell-Bloch equations [15], for which the restriction $\sigma < 0$ appears naturally. Thus we will allow the parameters in (1) to assume non-positive values. Because of its rich dynamics, including chaotic behavior, the Lorenz system has attracted much attention and many articles have been published that study it from a dynamical point of view.

In this work we are interested in the study of the bifurcation of small amplitude limit cycles from the *Hopf singularities* of (1), by which we mean singularities at which the linear part of the associated generating vector field has one non-zero real and two purely imaginary eigenvalues. More particularly we are interested in the cyclicity of the Hopf points that are centers within some, hence every, center manifold, where, roughly speaking, the *cyclicity* of the point is the maximum number of limit cycles that can bifurcate from it under small perturbations within family (1). The reader can consult [16] for a more precise definition of cyclicity.

It is well known that the local dynamics of an analytic system near a Hopf singularity restricted to any center manifold must be of either focus or center type ([4]). We recall that the equilibrium is a *center* of (1) if all the orbits on some,

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hence every, local center manifold at the origin are periodic; otherwise it is called a *saddle-focus*.

In [1] the authors study the Hopf bifurcation at both the trivial singularity (the origin) and the non-trivial singularities $E_{\pm}$ that sometimes exist in the Lorenz system, in the case that the equilibrium in question is a saddle-focus. Specifically, they compute the order of the saddle-focus on any center manifold and show that bifurcations of codimension 2 and 3 can take place at the trivial and non-trivial singularities, respectively. They also present some conditions on the parameter space for which the Hopf singularities are centers on the center manifold. See also [2].

In [16] the center variety of a *generalized* Lorenz system is characterized and the center cyclicity is obtained for the singularity at the origin, perturbing it in such a way the origin remains a Hopf point, so that there is still a center manifold through the point.

The problem of the center cyclicity for plane polynomial systems is well investigated. An essential obstacle for applying the two-dimensional theory to the case of higher dimensional systems, however, is the possible non-analyticity of the center manifold. This is the main reason why the techniques in [1] cannot be used to analyze the center cyclicity, but work only in a study of degenerate Hopf bifurcations of finite codimension. The work [16] overcomes this difficulty and presents a generalization of the theory to the case of three-dimensional systems.

The goal of this work is to complete the bifurcation study started in the papers [1] and [16] and to fill the gap that exists concerning the cyclicity of the centers on center manifolds. We analyze the number of limit cycles that can be made to bifurcate from a center at the non-trivial singularities as well as from a center at the origin, both under perturbation within the parameter space that maintains the character of the singularity as a Hopf point and under perturbation within the full family (1). The former restriction is natural since then a center manifold continues to exist to contain any limit cycle produced. In the second case, however, the singularity is allowed to become hyperbolic and the center manifold disappears. Our main result is the following, whose proof will be given in Section 3, applying the novel techniques developed in [16] which are generalized in Section 2.2.

Theorem 1. *For the Lorenz family* (1)*, with the exception of the system that corresponds to the parameter choice* $(\sigma, \rho, b) = (-1, -3 - 3\sqrt{2}, -2)$ *(for which the origin is not a center),*

1. *the cyclicity of any center (at either the origin or $E_{\pm}$), under perturbation within family that maintains the nature of the singularity as a Hopf point, is zero; and*
2. *the cyclicity of any center (at either the origin or $E_{\pm}$), under perturbation within full family* (1)*, is one.*

With regard to the system corresponding to the exceptional parameter string $(\sigma, \rho, b) = (-1, -3 - 3\sqrt{2}, -2)$, in Appendix B we derive a lower bound of one on the cyclicity of centers at the remote points $E_{\pm}$ under perturbation that preserves their identities as Hopf points, and a lower bound of two under general perturbation within family (1).

The Lü family [20] is given by

$$\dot{x} = A(y - x), \quad \dot{y} = Cy - xz, \quad \dot{z} = -Bz + xy, \tag{2}$$

and the Chen family [9] by

$$\dot{x} = A(y - x), \quad \dot{y} = (C - A)x + Cy - xz, \quad \dot{z} = -Bz + xy, \tag{3}$$

with parameters $(A, B, C) \in \mathbb{R}^3$. After a linear scaling the Chen and Lü families reduce (generically) to special cases of the Lorenz family (see [1], [2], and Section 4). From the results on the Lorenz system we can then derive as a byproduct the cyclicity of the centers for Chen and Lü systems. The result is the following theorem, whose proof is given in Section 4.

Theorem 2. *For both the Lü family* (2) *and the Chen family* (3)*,*

1. *the cyclicity of any center (either the origin or remote), under perturbation within family that maintains the nature of the singularity as a Hopf point, is zero; and*
2. *the cyclicity of any center (either the origin or remote), under perturbation within full family* (1)*, is one.*

This paper is organized as follows. Section 2 is devoted to a summarization and generalization of the mathematical tools needed for the analysis of the bifurcations that we will carry out later. In Section 3 we analyze the bifurcation that can take place at a center on a center manifold at any Hopf singularity in the Lorenz family. In the last section the results concerning the center cyclicity of Chen and Lü systems are derived.

By way of notation, the affine variety ([13], [21]) of an ideal $I = \langle f_1, \ldots, f_k \rangle$ in a polynomial ring will be denoted $\mathbf{V}(I)$ or $\mathbf{V}(f_1, \ldots, f_k)$. The generating vector field corresponding to a system of differential equations under consideration will be denoted $\mathfrak{X}$.

2. Centers in $\mathbb{R}^3$ and Their Cyclicity

In this section we summarize some of the results of [16] that we will need and extend them from the setting of perturbing a center within (4) below to that of perturbing a center in (6) (for which of course $\alpha = 0$) within the family (6).

2.1. **The center variety.** Any analytic family of differential equations on $\mathbb{R}^3$ having an isolated Hopf singularity can be transformed, by means of an affine change of coordinates that moves the singularity to the origin and a rescaling of time, into the form

$$\dot{x} = -y + \mathcal{F}_1(x, y, z; \mu), \quad \dot{y} = x + \mathcal{F}_2(x, y, z; \mu), \quad \dot{z} = \lambda z + \mathcal{F}_3(x, y, z; \mu), \tag{4}$$

where $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ contain only nonlinear terms and the parameter space is some set $E \subset \{(\lambda, \mu) : (\lambda, \mu) \in \mathbb{R}^* \times \mathbb{R}^p\}$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

By the *Lyapunov Center Theorem* (see a proof in [8]) it is known that a singularity of (4) at the origin is a center on the center manifold if and only if it admits a local real analytic (or, by [14], merely formal) first integral H of the form $H(x, y, z) = x^2 + y^2 + \cdots$, where the dots denote higher order terms. In particular by a well-documented algorithmic procedure (which can be found described, for example, in [10, §8.3]) we recursively compute coefficients in a formal series $H(x, y, z) = x^2 + y^2 + \cdots$ so that

$$\mathfrak{X}(H) = \sum_{j \geqslant 2} \tilde{\eta}_j(\lambda, \mu)(x^2 + y^2)^j. \tag{5}$$

The first non-vanishing coefficient gives the asymptotic stability or instability of the focus at the origin on any center manifold, and the origin is a center for a member of family (4) if and only if all the coefficients vanish on the corresponding parameter string.

The structure of the problem implies that the coefficient functions $\tilde{\eta}_j$, the so-called the *focus quantities*, are elements of $\mathbb{R}(\lambda)[\mu]$, the Noetherian ring of polynomials in μ with coefficients in the field of rational expressions in λ with real coefficients, hence can be written

$$\tilde{\eta}_j(\lambda,\mu) = \eta_j(\lambda,\mu)/d_j(\lambda), \quad \eta_j \in \mathbb{R}[\lambda,\mu] \text{ and } d_j \in \mathbb{R}[\lambda],$$

where in fact $d_j(\lambda) \neq 0$ for $\lambda \in \mathbb{R}^*$(see for example [14], [16]).

Thus writing $\mathcal{I} = \langle \eta_j : j \in \mathbb{N} \rangle$ for the ideal in $\mathbb{R}[\lambda,\mu]$ generated by the polynomial numerators of the focus quantities, the set of parameters that yield systems (4) with a center at the origin is the intersection of the real affine variety of $\mathcal{I}$, $V_{\mathscr{C}} \stackrel{\text{def}}{=} \mathbf{V}(\mathcal{I})$, in $\mathbb{R}^{p+1}$, with the set E of admissible parameters. In summary, the origin of system (4) with $(\lambda,\mu) = (\lambda^*,\mu^*)$ is a center if and only if $(\lambda^*,\mu^*) \in V_{\mathscr{C}} \cap E$.

In order to capture the full range of perturbations of Hopf points in the Lorenz family (1), starting with a member of family (4) (with parameter space $E \subset \mathbb{R}^* \times \mathbb{R}^p$) having a center at the origin, we will also be concerned with perturbations that take place in the larger family associated with (4),

$$\dot{x} = \alpha x - y + \mathcal{F}_1(x,y,z;\mu), \ \dot{y} = x + \alpha y + \mathcal{F}_2(x,y,z;\mu), \ \dot{z} = \lambda z + \mathcal{F}_3(x,y,z;\mu), \tag{6}$$

with parameter set $E' = E \times \mathbb{R} \subset \{(\lambda,\mu,\alpha) : (\lambda,\mu,\alpha) \in \mathbb{R}^* \times \mathbb{R}^p \times \mathbb{R}\}$. The theory described for family (4) holds for family (6), but now the coefficients $\tilde{\eta}_j$ in (5) depend on α, are analytic in the parameters but are not necessarily polynomials, and the sum in (5) starts at $j = 1$, with $\tilde{\eta}_1(\lambda,\mu,\alpha) = 2\alpha$.

2.2. **Estimating center cyclicity.** We now describe methods for bounding and estimating the cyclicity of a center at the origin of polynomial families in $\mathbb{R}^3$. Their generalization from perturbation of centers in family (4) to perturbation of centers in (6) are new. We note at the outset that because the set $\alpha = 0$ is a variety, $\mathbf{V}(\alpha)$, the set of centers in the family (6) is still picked out by a variety, namely $V_{\mathscr{C}} \cap \mathbf{V}(\alpha)$, upon its intersection with E'.

The theory we describe is presented in detail in Section 3 of [16] for the case that the perturbations are restricted to family (4). We introduce a polar-directional blow-up $\Phi : S^1 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$ defined by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = rw, \tag{7}$$

which because of the factor r in the z equation blows up the origin $(x,y,z) = (0,0,0)$ to the set $\{(\theta,r,w) : r = 0\}$. Elements of family (6) are transformed into systems of the form

$$\begin{aligned} \dot{\theta} &= 1 + \Theta(\theta,r,w;\mu) \\ \dot{r} &= \alpha r + \mathcal{R}(\theta,r,w;\mu) \\ \dot{w} &= (\lambda - \alpha)w + \mathcal{W}(\theta,r,w;\mu), \end{aligned} \tag{8}$$

which are analytic on a neighborhood of the invariant set $r = 0$. Using θ as an independent variable we replace (8) by

$$\frac{dr}{d\theta} = R(\theta,r,w;\mu), \quad \frac{dw}{d\theta} = (\lambda - \alpha)w + W(\theta,r,w;\lambda,\mu) \tag{9}$$

on some cylinder $C = \{(\theta, r, w) : |r| \leqslant \hat{r}, w \in M\} \subset S^1 \times \mathbb{R} \times \mathbb{R}$, where $\hat{r} > 0$ is sufficiently small and M is an arbitrary compact neighborhood of 0 in $\mathbb{R}$. It is apparent that by means of the transformation (7) for any $\tau > 0$ there is a one-to-one correspondence between 2π-periodic solutions of (9) in a neighborhood of the cylinder $r = 0$ within the region $\{(\theta, r, w) : |w| < \sqrt{\tau}\}$ and periodic orbits of (6) in a neighborhood of the origin $(x, y, z) = (0, 0, 0)$ and lying wholly outside the solid cone $C_\tau = \{(x, y, z) : z^2 \geqslant \tau(x^2 + y^2)\}$, since (6) is analytically conjugate to (8) on $\mathbb{R}^3 \setminus C_\tau$.

In a process described in Section 3 of [16] and in more detail in [7], by means of the analytic Poincaré first return map associated to (9) there is defined the analytic displacement map $d(r_0, w_0; \lambda, \mu, \alpha)$ from which by means of the Implicit Function Theorem we ultimately obtain a reduced displacement function

$$d_{\mathrm{red}}(r_0; \lambda, \mu, \alpha) = \sum_{j \geqslant 1} v_j(\lambda, \mu, \alpha) r_0^j \tag{10}$$

around $r = 0$, whose zeros correspond to periodic orbits of (6) that lie in a neighborhood of $(x, y, z) = (0, 0, 0)$ but wholly outside the solid cone C_τ. The discussions in [7] and in [16] are for family (4) but apply equally well in the setting of family (6). In the more restricted setting the sum in (10) starts at $j = 3$ and the functions v_j, which we term the *Poincaré-Lyapunov quantities*, have a polynomial character like that of the $\tilde{\eta}_j$, to which they bear an intimate relationship (described in Remark 9 below). In the general setting they are merely analytic; a straightforward computation yields

$$v_1(\lambda, \mu, \alpha) = e^{2\pi\alpha} - 1. \tag{11}$$

In the setting of family (4) all limit cycles that bifurcate from the center at the origin must lie in some (hence every) center manifold at the origin, every one of which is tangent to zero-eigenspace, the xy-plane. Clearly then the reduced displacement function must capture them all. The following proposition asserts that in the setting of (6), for which the singularity at the origin can become hyperbolic and center manifolds no longer persist, it is still true that all isolated periodic orbits that bifurcate from a center of (4) at the origin can be located by means of the positive zeros of the reduced displacement function (10) in a neighborhood of $r_0 = 0$.

Proposition 3. *Suppose the system* (4) *corresponding to parameter value* $(\lambda, \mu) = (\bar{\lambda}, \bar{\mu})$*, equivalently system* (6) *corresponding to parameter value* $(\lambda, \mu, \alpha) = (\bar{\lambda}, \bar{\mu}, 0)$*, has a center on the (therefore unique) local center manifold at the origin. There exists a neighborhood N of the origin in $\mathbb{R}^3$ and a neighborhood P of* $(\lambda, \mu, \alpha) = (\bar{\lambda}, \bar{\mu}, 0)$ *in E' such that for any parameter value in P, all periodic orbits of the corresponding system* (6) *that lie wholly within N correspond to zeros of the reduced displacement function d_{red} on a neighborhood of $r_0 = 0$.*

Proof. We will prove this proposition using the theory of invariant manifolds as expounded in Theorem 4.1 of [10] and its proof, whose notation we adopt in this discussion, exploiting the fact that the spectral gap that exists for the original system continues to exist for all sufficiently small perturbations within family (6), giving uniform estimates. In a nutshell, we show that under perturbation the normally hyperbolic local center manifold persists as (or, is replaced by) a normally hyperbolic invariant two-manifold through $(0, 0, 0)$ that also lies outside the cone

C_1 (except at the origin) and contains all periodic orbits of the perturbed system near the origin.

Without loss of generality we may assume that $\bar{\lambda} > 0$.

Choose a and b such that $0 < a < b < \bar{\lambda}$. Then the interval $[a, b]$ forms a spectral gap for the original system (4) with $(\lambda, \mu, \alpha) = (\bar{\lambda}, \bar{\mu}, 0)$ and for every system in family (6) with parameter (λ, μ, α) in a sufficiently small neighborhood P_1 of $(\bar{\lambda}, \bar{\mu}, 0) \in \mathbb{R}^* \times \mathbb{R}^p \times \mathbb{R}$. There thus exists a constant $K > 0$ such that for every parameter value in P_1, the estimates

$$||e^{tS}\xi|| \leqslant Ke^{at}||\xi|| \quad \text{and} \quad ||e^{-tU}\nu|| \leqslant Ke^{-bt}||\nu||$$

hold for the corresponding system (6) for all $t \geqslant 0$, where $S = \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}$, $U = \begin{pmatrix} \lambda \end{pmatrix}$, $\xi \in \mathbb{R}^2$, and $\nu \in \mathbb{R}$. But then for any choice of a constant $\rho > 0$ and for $\delta > 0$ satisfying

$$\delta < \left(\frac{b-a}{K}\right) \min\left\{\frac{\rho}{(1+\rho)(K+\rho)}, \frac{1}{2(K+1+\rho)}\right\}, \tag{12}$$

if a system (6) corresponding to parameter value $(\lambda, \mu, \alpha) \in P_1$ is such that

$$||(\mathcal{F}_1, \mathcal{F}_2)||_1 < \delta \quad \text{and} \quad ||\mathcal{F}_3||_1 < \delta \tag{13}$$

(supremum norms on all of $\mathbb{R}^3$) then there exists a unique C^1 function $f : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto z = f(x, y)$, depending of course on (λ, μ, α), satisfying

$$f(0,0) = 0, \quad Df(0,0) = 0, \quad ||Df(\xi)||_0 < \infty \tag{14}$$

whose graph W is an invariant manifold for the system. Moreover f is Lipschitz with Lipschitz constant satisfying

$$\operatorname{Lip}(f) < \rho. \tag{15}$$

Fix $\rho < 1$ and a value of δ meeting condition (12). By (14) and (15) and this choice of ρ, for any system (6) with parameter in P_1, the graph of the corresponding f lies wholly in the complement of the cone $C_1 = \{(x, y, z) : z^2 \geqslant x^2 + y^2\}$, except at the origin.

Choose $r_1 > 0$ such that the reduced displacement function is defined and analytic on the ball B_{4r_1} of radius $4r_1$ about the origin. If we then choose a neighborhood $P \subset P_1$ of $(\bar{\lambda}, \bar{\mu}, 0)$ such that the estimates

$$\sup_{(x,y,z)\in B_{r_1}} ||D(\mathcal{F}_1, \mathcal{F}_2)(x, y, z)|| < \frac{\delta}{3}, \qquad \sup_{(x,y,z)\in B_{r_1}} ||D\mathcal{F}_3(x, y, z)|| < \frac{\delta}{3} \tag{16}$$

hold for all choices of (λ, μ, α) in P, then as outlined in [10] there exists a C^∞ cut-off function γ that is identically 1 on $B_{r_1/3}$, identically 0 on $\mathbb{R}^3 \setminus B_{r_1}$, and such that multiplying the nonlinearities in (6) by γ, the analogues of condition (13) are met. This provides, for any perturbation with parameter in P, an invariant two-manifold W that lies (except at the origin) wholly in $\mathbb{R}^3 \setminus C_1$.

Note that by our choice of r_1 all periodic orbits wholly within $(\mathbb{R}^3 \setminus C_1) \cap B_{r_1}$ are in correspondence with zeros of the reduced displacement function d_{red}. Further shrinking r_1 only improves the estimates (16) and does not effect the existence of a suitable cut-off function, so we may shrink r_1 if necessary so that for any parameter in P, $W_{\text{loc}} := W \cap B_{r_1}$ is normally hyperbolic, like the local center manifold W^c_{loc} that it has replaced. Setting $N = B_{r_1}$, the normal hyperbolicity of W_{loc} implies that it contains every periodic orbit lying wholly in N. But then all periodic orbits that have bifurcated from the center of the original system correspond to zeros of the reduced displacement function, as was to be shown. □

For the next result concerning estimating cyclicity of centers we must first review some terminology.

Definition 4. Let $B = \{f_1, f_2, f_3, \dots\}$ be an ordered basis of an ideal I in a Noetherian ring. The *minimal basis* M of I with respect to B is the basis M of I defined by the following procedure:

(i) initially set $M = \{f_J\}$, where f_J is the first non-zero element of I;

(ii) sequentially check successive elements f_j, starting with $j = J+1$, and adjoin f_j to M if and only if $f_j \notin \langle M \rangle$, the ideal generated by M.

Thus for example the minimal basis of the principal ideal $\langle f \rangle$ with respect to the ordered basis $B = \{f^3, f^2, f\}$ is B itself, not the basis $\{f\}$ of minimal cardinality.

Our concern will be with the minimal basis of the ideal $\mathcal{B} = \langle v_j : j \in \mathbb{N} \rangle$ (the *Bautin* ideal) (or with the ideal $\mathcal{B}_m \overset{\text{def}}{=} \langle v_j : j \leqslant m \rangle$) in the Noetherian rings $\mathbb{R}(\lambda)[\mu]$, $\mathscr{G}_{(\lambda^*,\mu^*)}$, and $\mathscr{G}_{(\lambda^*,\mu^*,\alpha^*)}$, where the latter two are the rings of germs of analytic functions at (λ^*, μ^*) and $(\lambda^*, \mu^*, \alpha^*)$ in $\mathbb{R}^* \times \mathbb{R}^p$ and $\mathbb{R}^* \times \mathbb{R}^p \times \mathbb{R}$, respectively. To avoid an overabundance of notation we are using the same symbol v_j for what are really two different functions, $v_j(\lambda, \mu)$ and $v_j(\lambda, \mu, \alpha)$, and the same symbol $\mathcal{B}$ for what are really three different ideals. Similarly we will not make a notational distinction between a function v_j and the germ of which it is a representative. In every case we assume that bases of ideals are ordered by the increasing order of the indices of the generators.

The result from [16] which we wish to recall and generalize is Theorem 6 below, which was inspired by the work of Colin Christopher ([12]). Before stating it we must make an important observation, which we number for future reference.

Remark 5. The set $M = \{v_{j_1}(\lambda, \mu), \dots, v_{j_\kappa}(\lambda, \mu)\}$ used in defining the mapping F_κ of the next paragraph is the minimal basis of the ideal $\mathcal{B}_{j_\kappa} = \langle v_3, \dots v_{j_\kappa} \rangle$ *in the polynomial ring* $\mathbb{R}(\lambda)[\mu]$, but the condition on the rank of $d_P F_\kappa$ in Theorems 6 and 8 below has to do with the functions v_j as analytic mappings. Moreover in the proof of these two theorems the mappings v_j are always treated as analytic functions and the reference to M in the proof as a minimal basis of $\mathcal{B}_{j_\kappa}$ is with reference to that ideal in the ring of germs $\mathscr{G}_P$. *A priori* these minimal bases need not agree: the minimal basis of the ideal $\mathcal{B}_{j_\kappa} = \langle v_3, \dots v_{j_\kappa} \rangle$ in $\mathbb{R}(\lambda)[\mu]$ might strictly contain the minimal basis of the ideal $\mathcal{B}_{j_\kappa} = \langle v_3, \dots v_{j_\kappa} \rangle$ in $\mathscr{G}_P$. However, using the fact that the point P of interest yields a system with a center, it can be directly verified that in such a case the hypothesis in the two theorems that $d_P F_\kappa$ have maximal rank cannot hold. (The key idea is that if the minimal bases disagree then for some $r \in \{1, \dots, \kappa - 1\}$, $v_{j_{r+1}} = \varphi_1 v_{j_1} + \cdots + \varphi_r v_{j_r}$ for $\varphi_1, \dots, \varphi_r \in \mathscr{G}_P$, and row reduction of $d_P F_\kappa$ yields a matrix whose $(r+1)$st row evaluates to all zeros at P.) Thus in any case in which the two theorems can be applied the minimal bases in the two rings agree, which has the practical consequence that computations can always be done in the polynomial context, for which the tools of computational algebra like SINGULAR are available.

In the context of all bifurcations being restricted to family (4) (the setting of [16]), for any natural number κ up to the Bautin depth (the cardinality of the minimal basis) of $\mathcal{B} \subset \mathbb{R}(\lambda)[\mu]$ define a real analytic mapping $F_\kappa : \mathbb{R}^* \times \mathbb{R}^p \to \mathbb{R}^\kappa$

by

$$F_\kappa(\lambda,\mu) = (v_{j_1}(\lambda,\mu), \dots, v_{j_\kappa}(\lambda,\mu)), \tag{17}$$

where $\{v_{j_1}(\lambda,\mu), \dots, v_{j_\kappa}(\lambda,\mu)\}$ is the minimal basis of the ideal $\mathcal{B}_{j_\kappa} = \langle v_3, \dots v_{j_\kappa}\rangle$ in $\mathbb{R}(\lambda)[\mu]$. We denote by $d_P F_\kappa$ the $\kappa \times (p+1)$ Jacobian matrix of F_κ evaluated at $P \in E$. Recall that $V_{\mathscr{C}} \subset \mathbb{R}^{p+1}$ denotes the variety in the parameter space such that an element of the family (4) corresponding to (λ,μ) has a center at the origin if and only if $(\lambda,\mu) \in V_{\mathscr{C}} \cap E$. (With regard to smooth points of an affine variety consult Section 6 in Chapter 9 of [13].)

Theorem 6 ([16, Theorem 32]). *Let κ, $\{v_{j_1}, \dots, v_{j_\kappa}\}$, and F_κ be as in the preceding paragraph. Let C be an irreducible component of the center variety $V_{\mathscr{C}}$ associated to the origin of family* (4). *Suppose $\kappa \leqslant p+1$ and let $P = (\lambda^*,\mu^*) \in C \cap E$ be a point such that* $\operatorname{rank}(d_P F_\kappa) = \kappa$, *i.e., is maximal. Then the following holds:*

(i) There exists a neighborhood U of P in $\mathbb{R}^{p+1}$ such that $C \cap U$ is a submanifold of $\mathbb{R}^{p+1}$ of codimension at least κ and there exist bifurcations of (4) *producing $\kappa - 1$ small amplitude limit cycles from the origin for parameter values with (λ,μ) sufficiently close to P.*

(ii) If moreover $\operatorname{codim}(C) = \kappa$ *then P is a smooth point of C and the cyclicity of P and also of any point in a relatively dense open subset of C is exactly $\kappa - 1$.*

To prove the extension of this result to perturbations of centers in the larger family (6) we will need the following lemma.

Lemma 7. *Fix families* (4) *and* (6) *with their respective parameter sets E and $E' = E \times \mathbb{R}$. Suppressing the dependence of all the functions involved on λ and μ, let $\{v_k(\alpha)\}_{k=1}^{\infty}$ denote (temporarily, for the purpose of clarity of exposition) the Poincaré-Lyapunov quantities for family* (6) *and let $\{v_k\}_{k=3}^{\infty}$ be the Poincaré-Lyapunov quantities just for family* (4). *Fix (λ^*,μ^*) in E and suppose $\{v_{k_1}, \dots, v_{k_m}\}$ is the minimal basis of the ideal $\langle v_3, v_4, \dots\rangle \subset \mathscr{G}_{(\lambda^*,\mu^*)}$. Then $\{v_1(\alpha), v_{k_1}, \dots, v_{k_m}\}$ is the minimal basis of the ideal $\langle v_1(\alpha), v_2(\alpha), v_3(\alpha), \dots\rangle \subset \mathscr{G}_{(\lambda^*,\mu^*,0)}$.*

Proof. This lemma is a direct analogue of Lemma 6.2.8 of [21], whose proof goes through simply by making the obvious changes in notation. □

Theorem 8. *Let κ, $\{v_{j_1}, \dots, v_{j_\kappa}\}$, and F_κ be as in the paragraph preceding Theorem 6. Let C be an irreducible component of the center variety $V_{\mathscr{C}} \subset \mathbb{R}^{p+1}$ associated to the origin of family* (4). *Suppose $\kappa \leqslant p+1$ and let $P = (\lambda^*,\mu^*) \in C \cap E$ be a point such that* $\operatorname{rank}(d_P F_\kappa) = \kappa$, *i.e., is maximal. Then letting C also denote the set $\{(\lambda,\mu,0) : (\lambda,\mu) \in C\}$ in $\mathbb{R}^{p+2}$, the following holds:*

(i) There exists a neighborhood U' of $P' = (\lambda^*,\mu^*,0) \in C \cap E' \subset \mathbb{R}^{p+2}$ such that $C \cap U'$ is a submanifold of $\mathbb{R}^{p+2}$ of codimension at least $\kappa + 1$ and there exist bifurcations of (6) *producing κ small amplitude limit cycles from the origin for parameter values with (λ,μ,α) sufficiently close to P'.*

(ii) If moreover the codimension of C in $\mathbb{R}^{p+2}$ is $\kappa + 1$ then P' is a smooth point of C and the cyclicity of P' and also of any point in a relatively dense open subset of C is exactly κ.

Proof. For the sake of brevity we refer the reader to the proof of Theorem 6 in [16] and point out why it carries over to the situation of Theorem 8.

In analogy with F_κ define a real analytic mapping $F'_\kappa : \mathbb{R}^* \times \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^\kappa$ by

$$F'_\kappa(\lambda, \mu, \alpha) = (v_1(\lambda, \mu, \alpha), v_{j_1}(\lambda, \mu), \dots, v_{j_\kappa}(\lambda, \mu)). \tag{18}$$

Then by (11), $d_{P'}F'_\kappa$ is the $(\kappa + 1) \times (p + 2)$ matrix

$$\left(\begin{array}{ccc|c} 0 & \cdots & 0 & 2\pi \\ \hline & & & 0 \\ & d_P F_\kappa & & \vdots \\ & & & 0 \end{array}\right),$$

which by the hypothesis on κ and F_κ has maximal rank, $\kappa + 1$. Thus the first paragraph of the proof of Theorem 6 in [16] carries over to yield the first assertion in point (i).

At the level of germs $\{v_{j_1}(\lambda, \mu), \dots, v_{j_\kappa}(\lambda, \mu)\}$ is the minimal basis of the ideal $\mathcal{B}_{j_\kappa} = \langle v_3(\lambda, \mu), \dots v_{j_\kappa}(\lambda, \mu)\rangle$ in the ring $\mathscr{G}_{(\lambda^*, \mu^*)}$ (see the end of Remark 5). By Lemma 7, $\{v_1(\lambda, \mu, \alpha), v_{j_1}(\lambda, \mu), \dots, v_{j_\kappa}(\lambda, \mu)\}$ is the minimal basis of the ideal $\mathcal{B}'_{j_\kappa} = \langle v_1, \dots v_{j_\kappa}\rangle$ in $\mathscr{G}_{(\lambda^*, \mu^*, \alpha^*)}$. Filling out the basis of $\mathcal{B}'_{j_\kappa}$ to the minimal basis of $\mathcal{B} = \langle v_j : j \in \mathbb{N}\rangle$ in $\mathscr{G}_{(\lambda^*, \mu^*, \alpha^*)}$ (possible since by hypothesis κ is less than the Bautin depth of $\mathcal{B}$) and using the Domain Straightening Theorem as in the proof in [16] we find that we can cause κ positive zeros of the reduced displacement function to appear under suitable perturbation, completing the proof of part (i) of the theorem.

The proof of part (ii) follows exactly along the lines of the proof of Theorem 6 in [16]. □

Remark 9. In [16, §4] it is shown that for family (4), $v_3 = \pi\widetilde{\eta}_2$ and for $k \geqslant 2$, $v_{2k} \in \langle\tilde{\eta}_2, \dots, \tilde{\eta}_k\rangle$ and $v_{2k+1} - \pi\widetilde{\eta}_{k+1} \in \langle\tilde{\eta}_2, \dots, \tilde{\eta}_k\rangle$. Thus Theorems 6 and 8 could have been equally expressed in terms of the $\tilde{\eta}_j$. Because the focus quantities $\tilde{\eta}_k$ can be algorithmically computed our application of Theorem 8 will be expressed in terms of them.

3. Center Cyclicity of the Equilibria in the Lorenz Family

Returning to family (1), we repeat that we allow the parameters (ρ, σ, b) to take non-positive values, but with the one restriction that $b\,\sigma \neq 0$ else no singularity of (1) is isolated. The origin is an isolated singularity for all values of the parameters. Precisely when $b\,(\rho - 1) > 0$ there also exists the symmetrically located pair of remote singularities

$$E_\pm := (\pm\sqrt{b(\rho - 1)}, \pm\sqrt{b(\rho - 1)}, \rho - 1),$$

which are isolated when they exist. Since (1) is invariant under the involution $(x, y, z) \mapsto (-x, -y, z)$ it is sufficient to study only E_+ in order to obtain full information on both E_+ and E_-.

By Theorem 37 of [16], the origin is a Hopf point for (1) if and only if $\sigma = -1$ and $\rho > 1$ (and the eigenvalues of the linear part of the corresponding vector field $\mathfrak{X}$ are $-b \neq 0$ and $\pm\sqrt{\rho - 1}\,i$). It is a center if and only if in addition $b = -2$.

It is readily verified (and is already shown in [1]) that the singularities $E_\pm$ are Hopf points if and only if

$$b(\rho + \sigma) > 0 \quad \text{and} \quad b = \frac{-\sigma^2 - (3 - \rho)\sigma - \rho}{(\sigma + \rho)}, \tag{19}$$

and in this case the complex eigenvalues are $\pm\sqrt{b(\rho+\sigma)}\,i$ and the real eigenvalue is $-(b+\sigma+1)$, which cannot be zero without violating the condition $\sigma \neq 0$ for isolated singularities. These conditions (together with the condition $\sigma \neq 0$ for all singularities to be isolated) imply that $E_\pm$ are Hopf points only if (ρ, σ) lies in the region

$$\begin{aligned} H_E \stackrel{\text{def}}{=} & \{(\sigma,\rho) : -\sigma^2 - (3-\rho)\sigma - \rho > 0, \sigma \neq 0\} \\ & \cap [\{(\sigma,\rho) : \sigma + \rho < 0\} \cup \{(\sigma,\rho) : \rho - 1 > 0\}], \end{aligned}$$

which is the set of points in the (σ, ρ)-space (with $\sigma \neq 0$) between the two branches of the hyperbola and either above the line $\rho = 1$ or below the line $\sigma + \rho = 0$. The set of parameters (σ, ρ, b) for which $E_\pm$ are Hopf points is thus an open surface S in $\mathbb{R}^3$, in fact the portion of the graph of the function $b = \frac{-\sigma^2-(3-\rho)\sigma-\rho}{\sigma+\rho}$ that lies over the set H_E above. Note that S is thus

$$S = \mathbf{V}((\sigma+\rho)b + \sigma^2 + (3-\rho)\sigma + \rho) \cap \{(\sigma,\rho,b) \in \mathbb{R}^3 : (\sigma,\rho) \in H_E\}. \tag{20}$$

Hence in a study of the Hopf points of (1) we reduce to a two-dimensional parameter space S, with coordinates (σ, ρ) obtained by canonical projection $\pi(\sigma, \rho, b) = (\sigma, \rho)$ onto the subset H_E of the (σ, ρ)-plane.

3.1. **The canonical form at the nontrivial equilibria $E_\pm$.** Translating the nontrivial singularity E_+ of system (1) to the origin and placing the linear part of $\mathfrak{X}$ there in real Jordan canonical form, system (1) becomes, following the notation of [1],

$$\begin{aligned} \dot{x} &= -\omega_0 y + \frac{1}{C}(A_1 y^2 + A_2 z^2 + A_3(xy+xz) + A_5 yz) \\ \dot{y} &= \quad \omega_0 x + \frac{1}{C}(A_6 y^2 + A_7 z^2 + A_8(xy+xz) + A_{10} yz) \\ \dot{z} &= \quad az - \frac{1}{C}(A_6 y^2 + A_7 z^2 + A_8(xy+xz) + A_{10} yz) \end{aligned} \tag{21}$$

where

$$\begin{aligned} \omega_0 &= \sqrt{-(\sigma^2 + (3-\rho)\sigma + \rho)} > 0, \\ a &= \frac{2\sigma(1-\rho)}{\sigma+\rho} \neq 0, \text{ and} \\ C &= \frac{\rho^3(\sigma-1) + 5\rho^2(\sigma-1)\sigma - \rho\sigma^2(\sigma+15) - \sigma^2\left(\sigma^2+3\sigma-4\right)}{\sigma^2\sqrt{(\rho-1)(\rho+\sigma)^3}} \neq 0, \end{aligned} \tag{22}$$

and the values of the A_i are as given in Appendix A. The property $C \neq 0$ was proved in [1].

3.2. **The center problem at the nontrivial equilibria $E_\pm$.** The next result characterizes the set of admissible parameters for the family (1) for which $E_\pm$ exist and are centers. We will denote it $V_{\mathscr{C}}^E$, although strictly speaking it is not variety, but is the intersection of a variety with an open set.

Theorem 10. *Consider Lorenz family* (1) *with reduced two-dimensional parameter space S given by* (20)*, corresponding to the restriction to systems with remote singularities $E_\pm$ that are Hopf points.*

(i) The set of parameter values in S for which (1) *has centers at $E_\pm$ is* $V_{\mathscr{C}}^E = \mathbf{V}(\sigma+1) \cap S$.

(ii) $V_{\mathscr{C}}^{E}$ *is a codimension-one submanifold of* S*, hence is a codimension-two submanifold of the full parameter space* $\mathbb{R}^3$ *of* (1)*.*

Proof. Rather than work with system (1) directly we compute the first two focus quantities for system (21), for which the computations are much simpler. Expressed in terms of ω_0 and the original parameters σ and ρ the first non-zero focus quantity is

$$\tilde{\eta}_2 = \frac{\omega_0(\sigma,\rho)\,\eta_2(\sigma,\rho)}{d_2(\sigma,\rho)}, \tag{23}$$

where

$$\begin{aligned}
\eta_2(\sigma,\rho) &= (\sigma+1)(\rho+\sigma)(\rho^4+\rho^3\sigma+5\rho^3+\rho^2\sigma^2+21\rho^2\sigma+9\rho\sigma^3+35\rho\sigma^2+\\
&\quad 2\rho\sigma+6\sigma^4+15\sigma^3),\\
d_2(\sigma,\rho) &= 4(\rho-1)\left(\rho^3\sigma-\rho^3+2\rho^2\sigma^2-5\rho^2\sigma-\rho\sigma^3-9\rho\sigma^2-\sigma^4-3\sigma^3+\sigma^2\right)\\
&\quad \times C\sigma^2\sqrt{(\rho-1)(\rho+\sigma)^3}\in\mathbb{R}[\sigma,\rho]
\end{aligned}$$

and C is given by (22). The numerators η_3 and η_4 of the third and fourth focus quantities $\tilde{\eta}_3$ and $\tilde{\eta}_4$ have the form $(1+\sigma)(\rho+\sigma)^2N_3(\sigma,\rho)$ and $(1+\sigma)(\rho+\sigma)^3N_4(\sigma,\rho)$, where N_3 is a polynomial of degree 22 containing roughly 200 terms and N_4 is bigger, hence they are omitted. To solve the center problem we need only work with the numerators η_j of the focus quantities. We consider the polynomial ideal $\mathcal{I}_4=\langle\eta_2,\eta_3,\eta_4\rangle$ in the ring $\mathbb{R}[\sigma,\rho]$. Certainly $V_{\mathscr{C}}^{E}\subset\mathbf{V}(\mathcal{I}_4)=\mathbf{V}(\sqrt{\mathcal{I}_4})$, where $\sqrt{\mathcal{I}_4}$ denotes the radical of $\mathcal{I}_4$. In order to find the decomposition of $V_{\mathscr{C}}^{E}$ into a union of irreducible subvarieties as a means of proving the reverse inclusion, we employ the routine `minAssChar` in the `primdec.LIB` library of Singular to compute the prime decomposition of $\sqrt{\mathcal{I}_4}$, obtaining $\sqrt{\mathcal{I}_4}=\cap_{j=1}^{7}J_j$, where the prime ideals J_j are given by

1. $J_1=\langle g_1,g_2,g_3\rangle$, with generators
   (a) $g_1=223613\sigma^2-11804\rho-158445\sigma-236$
   (b) $g_2=223613\rho\sigma-65065\rho+432564\sigma+6504$
   (c) $g_3=223613\rho^2+951756\rho+2507553\sigma-42804$
2. $J_2=\langle\sigma-1,\rho+3\rangle$
3. $J_3=\langle 2\rho+7\sigma+3,9\sigma^2+2\sigma-3\rangle$
4. $J_4=\langle 13\sigma-8,13\rho+36\rangle$
5. $J_5=\langle 7\sigma+1,7\rho-3\rangle$
6. $J_6=\langle\sigma+\rho\rangle$
7. $J_7=\langle\sigma+1\rangle$.

We claim that $\mathbf{V}(J_j)\cap S=\varnothing$ for all $j\neq 7$. It is straightforward to see that the points $(\sigma,\rho)\in\mathbf{V}(J_j)$ with $j\neq 7$ do not meet the condition $b(\rho+\sigma)>0$ of (19). In short, we have proved that $\mathbf{V}(\mathcal{I}_4)\cap S=\mathbf{V}(J_7)\cap S$, hence $V_{\mathscr{C}}^{E}\subset\mathbf{V}(J_7)$.

To establish the reverse inclusion $\mathbf{V}(J_7)\subset V_{\mathscr{C}}^{E}$, we note that $V(x,y,z)=z-\frac{1}{2}x^2$ is actually an inverse Jacobi multiplier for (1) when $\sigma=-1$ (which by the second condition in (19) forces $b=-2$). This is true because V solves the partial differential equation $\mathfrak{X}(V)=V\,\mathrm{div}\mathfrak{X}$ where $\mathrm{div}(\cdot)$ denotes the divergence operator (see [5] for details). Then by Theorem 4 of [6] this implies that system (1) has a center at the origin when $\sigma=-1$. This establishes part (i) of the theorem.

A direct computation shows that the surfaces S and $\mathbf{V}(\sigma+1)$ intersect transversely, which establishes part (ii) of the theorem. $\square$

Remark 11. The work [2] finds the set of parameters giving centers as described in Theorem 10 (they show that the system restricted to the center manifold is Hamiltonian). This part of Theorem 10 is also implicit in [1]. Our proof is computationally different and our approach allows us to analyze the cyclicity of the centers. Indeed one can check that there are only two points in $\mathbf{V}(\mathcal{I}_3) \cap S$ such that η_4 is different from zero at those points. These two points corresponds with systems (1) whose saddle-foci at $E_\pm$ have (simultaneously) cyclicity three as was shown (independently) in [1] and [23].

3.3. **The center cyclicity of the nontrivial equilibria $E_\pm$.** Using Theorem 6 we have the following result for the Lorenz family.

Proposition 12. *Suppose $P \in V_{\mathscr{C}}^E = S \cap \mathbf{V}(\sigma + 1)$, S given by (20), so that the singularities $E_\pm$ exist and are centers (Theorem 10). Then when P is perturbed to a nearby point in S, i.e., the perturbation in the Lorenz family (1) is such that $E_\pm$ remain Hopf points, no limit cycle bifurcates from $E_\pm$, except for the single value $P = (\sigma, \rho, b) = (-1, -3 - 3\sqrt{2}, -2)$.*

Proof. Since the codimension of $V_{\mathscr{C}}^E$ with respect to the set of admissible perturbation parameters S is one (Proposition 10(ii)), we apply Theorem 6 with $\kappa = 1$. In accordance with Remark 9 we define, in analogy with (17), the map $H_2 : \mathbb{R}^2 \to \mathbb{R}$ given by $H_2(\sigma, \rho) = \tilde{\eta}_2(\sigma, \rho)$. Using (23), for an arbitrary point $P = (\sigma, \rho) = (-1, \rho) \in V_{\mathscr{C}}^E$ we directly compute the 1×2 matrix

$$d_P H_2 = \begin{pmatrix} \dfrac{(\rho^2 + 6\rho - 9)\sqrt{1 - \rho}}{4\sqrt{2}(\rho - 3)(\rho - 1)^2(2\rho - 3)} & 0 \end{pmatrix}. \tag{24}$$

which exists on S and has maximal rank 1 except for $\rho = -3 - 3\sqrt{2}$, since no parameter pair (σ, ρ) for $\rho \in \{-3 + 3\sqrt{2}, 3, 1, 3/2\}$ lies in S. The result follows from Theorem 6(ii). $\square$

Now we consider unrestricted perturbation within family (1) of centers at the remote singularities $E_\pm$. Notably we allow perturbations of the linear part of system (1) in such a way the singularity no longer need remain a Hopf point.

Proposition 13. *The cyclicity of centers located at the singularities $E_\pm$ of the Lorenz family (1) under unrestricted perturbation within the family (1) is one, except for original parameter choice $P : (\sigma, \rho, b) = (-1, -3 - 3\sqrt{2}, -2)$.*

Proof. Since there are no restrictions on the admissible parameters and the codimension of $V_{\mathscr{C}}^E$ with respect to the full $(\sigma, \rho, b)-$parameter space is two (Proposition 10(ii)), we apply Theorem 8 with $\kappa = 1$. The remainder of the proof is the same as that for Proposition 12, except that the result is a consequence of Theorem 8(ii). $\square$

Remark 14. The Lorenz family (1) restricted to the parameter values $b = 2\sigma$ possesses the invariant algebraic surface $F(x, y, z) = 2\sigma z - x^2 = 0$ (see for example [18]). In fact this surface becomes the center manifold of both equilibria $E_\pm$ in the center case, that is, when $b = 2\sigma = -2$ and $\rho < 1$. By [24] we know that when the invariant surface $F = 0$ exists then the Lorenz system has no limit cycle anywhere in the phase space $\mathbb{R}^3$. This means that the cyclicity of the centers at $E_\pm$ is zero under perturbation within family (1) but with parameters restricted to $S \cap \{(\sigma, \rho, b) : b = 2\sigma\}$. Proposition 12 shows that the cyclicity of centers at $E_\pm$ is still zero under perturbation restricted to the somewhat less confining parameter set S, and thereby allowing the breaking of the invariant surface $F = 0$. Proposition 13 shows that when parameters are completely unrestricted the cyclicity increases to one.

3.4. **The center cyclicity of the origin.** The *generalized* Lorenz system is the four-parameter family of quadratic differential equations on $\mathbb{R}^3$ given by

$$\dot{x} = \sigma(y - x), \quad \dot{y} = \rho x + cy - xz, \quad \dot{z} = -bz + xy, \tag{25}$$

which reduces to the Lorenz family (1) when $c = -1$. In [16, §9.3] it is shown that the trivial singularity (the origin) is a Hopf point if and only if $\sigma = -1$ and $\rho - 1 > 0$ and that it is a center if and only if in addition $b = -2$ ([16, Theorem 37]). It is further shown that when the origin is a center then under any perturbation that maintains its status as a Hopf singularity no limit cycle can be produced: the cyclicity is zero ([16, Theorem 38]). The next result gives the exact number of limit cycles that can be made to bifurcate from a center at the origin of (1) under unrestricted small perturbation of the parameters in (1).

Proposition 15. *The cyclicity of a center located at the origin of the Lorenz family* (1)*, under perturbation within family* (1)*, is one.*

Proof. Since the subset of the parameter space that yields systems with a Hopf point at the origin is the codimension-one surface $S \stackrel{\text{def}}{=} \mathbf{V}(\sigma + 1) \cap \{(\sigma, \rho, b) : \rho - 1 > 0\}$ in the full three-dimensional parameter space and the center set for the origin is the variety $\mathbf{V}(\sigma + 1, b + 2)$ intersected with the same open set $\{(\sigma, \rho, b) : \rho - 1 > 0\}$, a codimension-two submanifold of the full parameters space, we apply Theorem 8 with $\kappa = 1$. As in the proof of Proposition 12 we define, in analogy with (17), a map $H_2 : \mathbb{R}^2 \to \mathbb{R}$ by

$$H_2(\rho, b) = \tilde{\eta}_2(-1, \rho, b) = -\frac{b + 2}{\rho(b^2 + 4\rho - 4)}.$$

Then $d_P H_2$, evaluated at an arbitrary point $P : (\rho, b) = (\rho, -2)$ in the center set, is the 1×2 matrix

$$d_P H_2 = \begin{pmatrix} 0 & \frac{1-\rho}{4\rho^3} \end{pmatrix},$$

which exists on S and has maximal rank 1, since $\rho = 1$ violates the condition for a Hopf point. The result follows from Theorem 8(ii). □

3.5. **Proof of Theorem 1.** As noted at the beginning of the previous subsection, it is shown in [16, Theorem 38] that when the origin is a center then under any perturbation that maintains its status as a Hopf singularity no limit cycle can be produced. This fact together with Proposition 12 gives the first assertion in Theorem 1. The second assertion is an immediate consequence of Propositions 13 and 15.

4. Center cyclicity in the Chen and Lü families

In this section we prove Theorem 2 using straightforward arguments based on Theorem 1 and the clever step pointed out in several papers (see for instance [1]), that asserts that the Chen and Lü families are, generically, particular subfamilies of the Lorenz family.

Proof of Theorem 2. If $C = 0$ in either the Lü family (2) or the Chen family (3) it is easily verified that when the singularities are isolated none can be a Hopf point. When $C \neq 0$ then for both families the same linear scaling in time and the state variables

$$(x, y, z, t) \mapsto (-x/C, -y/C, -z/C, -Ct)$$

transforms the Lü family (2) into the Lorenz family (1) with parameters $\sigma = -A/C$, $\rho = 0$ and $b = -B/C$ and transforms the Chen family (3) into the Lorenz family (1) with parameters $\sigma = -A/C$, $\rho = A/C - 1$ and $b = -B/C$, hence with the restriction $\rho + \sigma = -1$. In both cases it is impossible for (σ, ρ, b) to be the exceptional parameter string $(\sigma, \rho, b) = (-1, -3 - 3\sqrt{2}, -2)$ of Theorem 1. Thus Theorem 2 is an immediate consequence of Theorem 1. □

Appendix A. The Coefficients A_i in (21)

$$A_1 = -\frac{\rho^3\sigma - \rho^3 + 5\rho^2\sigma^2 - 7\rho^2\sigma - 3\rho\sigma^3 - 19\rho\sigma^2 + 2\rho\sigma - \sigma^4 - \sigma^3 + 8\sigma^2}{(\rho - 1)\sigma^2(\rho + \sigma)}$$

$$A_2 = \frac{2\left(2\rho^3\sigma - \rho^3 + 3\rho^2\sigma^2 - 10\rho^2\sigma - 4\rho\sigma^3 - 17\rho\sigma^2 + 4\rho\sigma - \sigma^4 + 8\sigma^2\right)}{C\sigma(\rho + \sigma)^3}$$

$$A_3 = -\frac{\omega_0\left(\rho^2\sigma - \rho^2 - 4\rho\sigma^2 - 8\rho\sigma - \sigma^3 + \sigma^2 + 4\sigma\right)}{(\rho - 1)\sigma^2(\rho + \sigma)}$$

$$A_5 = -\frac{\rho^5\sigma - \rho^5 + 3\rho^4\sigma^2 - 7\rho^4\sigma + 2\rho^3\sigma^3 - 10\rho^3\sigma^2 + 6\rho^2\sigma^4 - 6\rho^2\sigma^3}{(\rho - 1)\sigma^2(\rho + \sigma)^3} - \frac{16\rho^2\sigma^2 + 3\rho\sigma^5 + 16\sigma^3 - 29\rho\sigma^4 - 32\rho\sigma^3 + 8\rho\sigma^2 - \sigma^6 - 3\sigma^5 + 8\sigma^4}{(\rho - 1)\sigma^2(\rho + \sigma)^3}$$

$$A_6 = -\frac{\omega_0\left(\rho + \sigma^2 + 2\sigma\right)}{(\rho - 1)\sigma^2}$$

$$A_7 = -\frac{\omega_0(\rho - 3\sigma - 4)}{C\sigma(\rho + \sigma)}$$

$$A_8 = \frac{\rho^2\sigma - \rho^2 - 2\rho\sigma + 2\rho + \sigma^3 + 3\sigma^2 + 4\sigma}{(\rho - 1)\sigma^2}$$

$$A_{10} = -\frac{\omega_0(\sigma + 1)\left(\rho^2 - 2\rho\sigma + \sigma^2 + 4\sigma\right)}{(\rho - 1)\sigma^2(\rho + \sigma)}.$$

Appendix B. A lower bound of two on the cyclicity of $E_\pm$ for the exceptional system

It is well known ([11, §5.1], [22]) that if the origin is a center for a system of the form (6) with $(\alpha, \lambda, \mu) = (0, \lambda^*, \mu^*) \in \mathbb{R} \times \mathbb{R}^* \times \mathbb{R}^p$ and we perturb the system along a smooth curve $\varepsilon \mapsto \gamma(\varepsilon) = (\alpha(\varepsilon), \lambda(\varepsilon), \mu(\varepsilon)) \subset \mathbb{R}^{p+2}$ with $\gamma(0) = (0, \lambda^*, \mu^*)$ in such a way that, for $|\varepsilon| \ll 1$,

$$|\alpha(\varepsilon)| \ll |\tilde{\eta}_2(\gamma(\varepsilon))| \ll |\tilde{\eta}_3(\gamma(\varepsilon))| \ll \cdots \ll |\tilde{\eta}_\ell(\gamma(\varepsilon))| \ll 1$$

with $\alpha(\varepsilon)\,\tilde{\eta}_2(\gamma(\varepsilon)) < 0$ and $\tilde{\eta}_j(\gamma(\varepsilon))\,\tilde{\eta}_{j+1}(\gamma(\varepsilon)) < 0$ for any $j = 2, \ldots, \ell - 1$, then $\ell - 1$ small amplitude limit cycles can be made to bifurcate from the center at the origin in family (6). We now use this fact to give lower bounds on the cyclicity of centers at the remote points $E_\pm$ for the Lorenz system corresponding to the exceptional paramter string $(\sigma, \rho, b) = (-1, -3 - 3\sqrt{2}, -2)$, for which $d_P H_2 = 0$ in (24) and the proof of Proposition 12 breaks down.

Consider any point $(\sigma^*, \rho^*) = (-1, \rho^*) \in V_{\mathscr{C}}^E$ (so with $\rho^* < 1$) and a curve $\varepsilon \mapsto \gamma(\varepsilon) = (\sigma(\varepsilon), \rho(\varepsilon)) \subset \mathbb{R}^3$ with $\gamma(0) = (\sigma^*, \rho^*)$. Assuming γ is sufficiently smooth at $\varepsilon = 0$ we have $\sigma(\varepsilon) = -1 + \sigma_1 \varepsilon + O(\varepsilon^2)$ and obtain

$$\tilde{\eta}_2(\gamma(\varepsilon)) = \frac{((\rho^*)^2 + 6\rho^* - 9)\sigma_1}{4\sqrt{2}(\rho^* - 3)(1 - \rho^*)^{3/2}(2\rho^* - 3)}\,\varepsilon + O(\varepsilon^2)$$

so that the special case $\rho^* = -3 - 3\sqrt{2}$ is the only admissible value of ρ^* that forces $\tilde{\eta}_2(\gamma(\varepsilon)) = O(\varepsilon^2)$ for any smooth γ with linear term $(\sigma_1 \neq 0)$.

A simple realization of one limit cycle retaining the nature of $E_\pm$ as Hopf points is obtained with the linear perturbation

$$\gamma(\varepsilon) = (\sigma(\varepsilon), \rho(\varepsilon)) = (-1 + \varepsilon, -3 - 3\sqrt{2} + 2\varepsilon), \tag{26}$$

for which

$$\tilde{\eta}_2(\gamma(\varepsilon)) = \xi_2 \varepsilon^2 + O(\varepsilon^3), \quad \tilde{\eta}_3(\gamma(\varepsilon)) = \xi_1 \varepsilon + O(\varepsilon^2)$$

with real constants $\xi_2 < 0 < \xi_1$.

We next perturb in such a way that $E_\pm$ are no longer Hopf points, and obtain one more limit cycle, hence a total of two. The singularity E_+ will have two complex eigenvalues with common real part α only if $\alpha(\sigma, \rho, b)$ satisfies the cubic equation

$$8\alpha^3 + 8(1+b+\sigma)\alpha^2 + 2[b^2 + (1+\sigma)^2 + b(2+\rho+3\sigma)]\alpha + b(\rho + b\rho + 3\sigma + b\sigma - \rho\sigma + \sigma^2) = 0.$$

Making the perturbation (26) for $(\sigma(\varepsilon), \rho(\varepsilon))$ and in addition the nonlinear perturbation $b(\varepsilon) = -2 + (5 - 3\sqrt{2})\varepsilon + (-55 + 39\sqrt{2})\varepsilon^2$, the resulting $\alpha(\varepsilon)$ solution of the cubic is $\alpha(\varepsilon) = \xi_3 \varepsilon^3 + O(\varepsilon^4)$ with a real constant $\xi_3 > 0$, yielding the additional limit cycle.

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